2.4 Convex Hull and Linear Combinations

In linear programming, we are given a description of the feasible set of points by a set of linear inequalities

$$P = \{x \in R_+^n : Ax \leq b\}$$

When we solve a linear program by the simplex method, issues such as the dimension of P and which inequalities are necessary for the description of P do not need to be addressed.

Integer programming is different as the sharpness of its **LP relaxation** directly impact on its solution procedure and therefore its difficulty.
Our aim in following sections is to seek a minimal linear inequality description of the **LP relaxation**.

Definition 2.4.8 Given a set $S \subseteq R^n$. A point $x \in R^n$ is a **convex combination** of points of S, if there exists a finite set of points $\{x_i\}_{i=1}^t \in S$ and a $\lambda \in R_+^t$ with $\sum_{i=1}^t \lambda_i = 1$, such that

$$x = \sum_{i=1}^{t} \lambda_i x^i$$

The **convex hull** of S, denoted by ***conv*(*S*)**, is the set of all points that are convex combinations of points in S.

Example1: Consider linear constraints:

$$21x_1 + 8x_2 \geq 28,$$

$$2x_1 - 2x_2 \leq 3,$$

$$7x_1 + 20x_2 \leq 70,$$

$$x_1, x_2 \in Z^+$$

Draw a diagram to show its constraints. Find all feasible integer solutions and therefore the **convex hull** of all feasible integer points. واجب

When the LP relaxation gives the convex hull of IP solutions. It is a sharp relaxation, and is known as the best one can achieve in terms of modelling.

Therefore, if we could formulate the IP model so that its constraints pass through feasible points without excluding any, then this IP can be solved as an LP and integrality will be automatically satisfied. We will see later that in principle this reformulation is always possible (provided the data of the IP is rational).

Remarks

• In contrast to IP models, MIP models are harder to represent. Parts of their solution is integer and will correspond to something like the graph, while the rest is real and will be on another dimension such as that perpendicular to the page. Solution will be on lines that pass through the dots parallel to the real dimension, say x_3.

• Generally speaking, finding an inequality description of *conv*(S) is not easy as it might lead to an exponential increase in the problem size. Therefore, questions such as the dimension of *conv*(S), the necessity of a certain inequality for the description of *conv*(S) and so on are very important in the solution of MIP/IP.

Before getting to the minimal description, we first recall some basic results (without proof) about space and dimensions from linear algebra.

Definition 2.4.1 A set of points $x_1, \ldots, x_k \in R^n$ is **linearly independent** if the unique solution of

$$\sum_{i=1}^{k} \lambda_i \; x_i = 0$$

Where $\lambda_i = 0, \forall i = 1, \ldots, k$.

• $(1,0,0)^T$, $(0,1,0)^T$, $(0,0,1)^T$ are obviously linearly independent.
• The maximum number of linearly independent points in R^n is n.

Definition 2.4.2 The maximum number of linearly independent rows (columns) of an $m \times n$ matrix A is the **rank** of A, denoted by $rank(A)$.

• Linear equations $Ax = b$ has solution iff $rank(A) = rank(A, b)$.

Affine Independence: The set $\{x_0, x_1, \dots, x_k\}$ is affinly independent if their affine hull is k-dimensional. Equivalently, $\{x_1 - x_0, \dots, x_k - x_0\}$ is linearly independent. نثبت نقطة لحساب الاستقلال هنا

Definition 2.4.3 A set of points $x_1, \dots, x_k \in R^n$ is **affinely independent** if the unique solution of

$$\sum_{i=1}^{k} \lambda_i \, x_i = 0, \quad \sum_{i=1}^{k} \lambda_i = 0$$

Where $\lambda_i = 0, \forall i = 1, \dots, k$.

Note: Linear independence implies affine independence, but the converse is not true. e.g.

$$\binom{1}{0}, \binom{2}{0}, \binom{1}{1}$$

are affinely independent but not linearly independent.

$$\binom{1}{1} - \binom{1}{0} = \binom{0}{1}, \quad \binom{2}{1} - \binom{1}{0} = \binom{1}{0}$$

So they are affinely independent where the two vectors $\binom{0}{1} and \binom{1}{0}$ are linearly independent.

• The maximum number of affinely independent points in R^n is $n + 1$:
n linearly independent points and the zero vector.

Proposition 2.4.4 If $x \in R^n$ and linear equations $Ax = b$ has solution, then the maximum number of affinely independent solutions to $Ax = b$ is $n + 1 - rank(A)$.

Example2:
Suppose

$$(A, b) = \begin{pmatrix} 1 & -4 & -3 \\ -2 & 8 & 6 \end{pmatrix}$$

How many affinely independent solutions are there to the linear system $Ax = b$?

SOLUTION: we have $\{x \in R^n : Ax = b\} \neq \phi$.
$rank(A) = rank(A, b) = 1$. By **Proposition 2.4.4**, the maximum number of affinely independent solutions of $Ax = b$ is $n + 1 - rank(A) = 2 + 1 - 1 = 2$.

Two such solutions are (5, 2) and (1, 1). See below

$$\begin{pmatrix}1 & -4\\ -2 & 8\end{pmatrix} = \begin{pmatrix}-3\\ 6\end{pmatrix} \rightarrow \begin{pmatrix}1 & -4\\ 0 & 0\end{pmatrix} = \begin{pmatrix}-3\\ 0\end{pmatrix} \rightarrow x_1 = -3 + 4x_2$$

Let x_2=1 then x_1=1, let x_2=2 then x_1=5 اواجمع المعادلتين

Definition 2.4.5 $H \subseteq R^n$ is a **subspace** if $\underline{x} \in H$ implies $\lambda x \in H$ for all $\lambda \in R$, and if $x, y \in H$ implies $x + y \in H$.

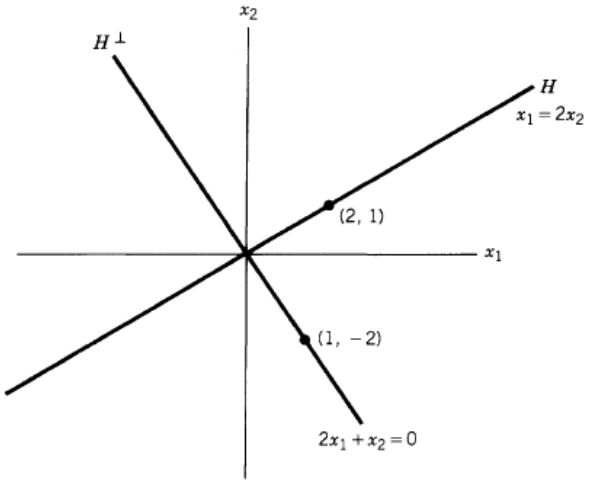
Proposition 2.4.6 $H \subseteq R^n$ is a subspace <u>if and only if</u> $\exists$ an $m \times n$ matrix A such that $H = \{x \in R^n : Ax = 0\}$.

Definition 2.4.7 If $H \subseteq R^n$ is a subspace, then

$$\{x \in R^n : x^T y = 0 \; for \; y \in H\}$$

is a subspace and is called the orthogonal subspace of H and is denoted by $H^{\perp}$.

Example3:
Justify that $H = \{x \in R^2 : x_1 - 2x_2 = 0\}$ and $H^{\perp} = \{x \in R^2 : 2x_1 + x_2 = 0\}$ are subspaces R^2. Are they orthogonal? Calculate their dimensions.



2.5 Polyhedra and Dimensions

Definition 2.5.1 A **polyhedron** $P \subseteq R^n$ is the set of points that satisfy a finite number of <u>linear inequalities</u>:

$$P = \{x \in R^n : Ax \leq b\},$$

Where A is an $m \times n$-matrix and $b \in R^m$ is a vector.

• A polyhedron is a **convex set**:
$\forall x_1, x_2 \in P$, their linear combination $\lambda x_1 1 + (1 - \lambda)x_2 \in P, \forall \lambda \in [0,1]$, and it is the intersection between a finite number of halfspaces.

• A rational polyhedron is one with matrix (A, b) having rational entries.
In later parts of this module we assume all polyhedra considered are ***rational****.*

Definition 2.5.2 A bounded polyhedron is called a **polytope**.

• A polyhedron $P \subseteq R^n$ is bounded if there exists $\omega \in R_+$ such that:

$P \subseteq \{x \in R^n : -\omega \leq x_j \leq \omega, \ j = 1, \ldots, n\}$.

Definition 2.5.3 $C \subseteq R^n$ is a **cone** if $x \in C$ implies $\lambda x \in C, \forall \lambda > 0$.

• The polyhedron $\{x \in R^n : Ax \leq 0\}$ is a cone.

Definition 2.5.4 A polyhedron $P \subseteq R^n$ is of **dimension** k, written as $dim(P) = k$, if the maximum number of affinely independent points in P is $k + 1$.

• A polyhedron $P \subseteq R^n$ is full-dimensional if $dim(P) = n$.

• If P is not full-dimensional then at least one of the inequalities (****) $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ is satisfied at equality by all points of P.

Exercise Calculate $dim(P)$ for each of the following polyhedra:
1. $P_1 = \{\underline{x} \in R^2 : x_1 + x_2 \leq 2, \ x_1 \geq 0\}$;
2. $P_2 = \{\underline{x} \in R^2 : x_1 + 2x_2 = 2\}$;
3. $P_3 = \{\underline{x} \in R^3 : x_1 = 10\}$;

Solution
1. $P_1 \subset R^2$, so $dim(P_1) \leq dim(R^2) = 2$;
Also observe that $(0,0)^T, (1,0)^T, (0,2)^T$ are three affinely independent points in P_1 (by $\sum_{i=1}^{k} \lambda_i \ \underline{x_i} = 0$). بما انه النقاط مستقلات خطيا عليه مستقلات افينيا

By **Definition 2.5.4** $dim(P_1) \geq 3 - 1 = 2$ (على الاقل 3 نقاط مستقلة افينيا); Combine the above, we get $dim(P_1) = 2$. Or (3=k+1=2+1 so k=2).
ممكن اخذ نقاط اخرى لكن يجب ان تكون affinely independent

2. $P_2 \subset R^2$ and having one (and only one) equality constraint, (by **Proposition 2.4.4)** we have the maximum number of affinely independent points $= n + 1 - rank(A) = 2 + 1 - 1 = 2$. So $dim(P_2) = 1$ (by **Definition 2.5.4)** . Where $(2,0)^T$, $(0,1)^T$ are two sample affinely independent points in P_2.

3. $P_3 \subset R^3$ and having one (and only one) equality constraint, (by **Proposition 2.4.4)** we have the maximum number of affinely independent points $= n + 1 - rank(A) = 3 + 1 - 1 = 3$. So $dim(P_2) = 2$ (by **Definition 2.5.4 we mean** 3=k+1=2+1 so k=2).

(Three sample affinely independent points are $(10,0,0)^T$, $(10,0,1)^T$, $(10,1,0)^T$.)

For polyhedron $P = \{x \in R^n : Ax \leq b\}$, let $M = \{1, 2, \dots, m\}$ and a_i be the ith row of A, define

• $M^{=} \triangleq \{i \in M : a_i\, x = b_i,\ \forall x \in P\}$;
• $M^{\leq} \triangleq \{i \in M : a_i\, x < b_i,\ for\ some\ x \in P\} = M \backslash M^{=}$;

• Let $(A^{=}, b^{=})$ represents the corresponding rows of A for equality constraints;

• Let $(A^{\leq}, b^{\leq})$ represents the corresponding rows of A for inequality constraints;

Then $P = \{x \in R^n : A^{=}x = b^{=},\ A^{\leq}x \leq b^{\leq}\}$.

• If $i \in M^{\leq}$, then $(a_i\, , b_i)$ cannot be written as a linear combination of the rows of $(A^{=},\ b^{=})$.

Definition 2.5.5 $x \in P$ is called an **inner point** of P if $a_i\, x < b_i,\ for\ all\ i \in M^{\leq}$. $x \in P$ is called an **interior point** of P if $a_i\, x < b_i$ for all $i \in M$.

• Every nonempty polyhedron $P = \{x \in R^n : Ax \leq b\}$ has an inner point.

• If $P \subset R^n$, then $dim(P) + rank(A^{=}, b^{=}) = n.$ (*)

• A polyhedron is full-dimensional <u>if and only if</u> it has an interior point. (**)

Example4: Suppose P ⊂ R3 is given by

$$\begin{array}{rcrcrcl} & x_1 & + & x_2 & + & x_3 & \leq & 1; \\ - & x_1 & - & x_2 & - & x_3 & \leq & -1; \\ & x_1 & & & + & x_3 & \leq & 1; \\ - & x_1 & & & & & \leq & 0; \\ & & - & x_2 & & & \leq & 0; \\ & & & & & x_3 & \leq & 2; \\ & x_1 & + & x_2 & + & 2x_3 & \leq & 2; \end{array}$$

1. Define $M^{=}$, $M^{\leq}$, $(A^{=}, b^{=})$ and $(A^{\leq}, b^{\leq})$;
2. Find $dim(P)$;
3. Is there any interior point?

Solution

1. Index constraints by the sequence in which they were given.
$M^{=} = \{1, 2\}$; $M^{\leq} = \{3, 4, \dots, 7\}$;

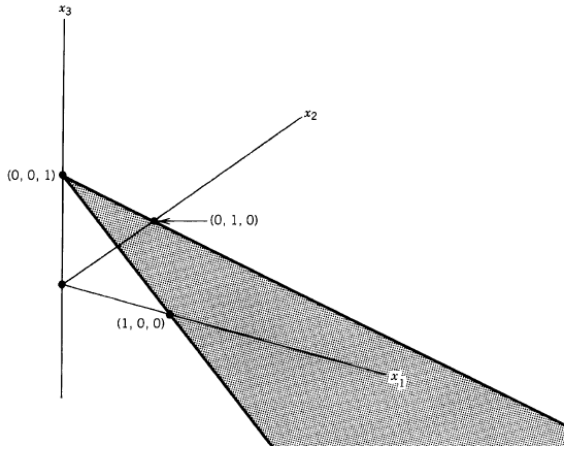
$$(A^{=}, b^{=}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}; (A^{\leq}, b^{\leq}) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}.$$

2. We know at least three points (1, 0, 0), (0, 1, 0), (0, 0, 1) lie in P which are affinely independent by 2.4.3. Hence by 2.5.4 dim (P) is at least k=2. So, $dim(P) \geq 2$

Also, constraints 1 and 2 mean all the points satisfy $x_1 + x_2 + x_3 = 1$. We have said that if P ⊂ R3, then $dim(P) = n - rank(A^{=}, b^{=}) = 3 - 1 = 2$, (*) (also, because all the points satisfy the equality, we have $rank\ (A^{=}, b^{=}) \geq 1$ and as its two rows are linearly independent). So by (*), $dim(P) \leq 2$. Therefore, $dim(P) = 2$.

More explanation, in our case we know at least one equality constraint, so by (****) it is not full dim. By (*) we have $dim(P) = n - rank(A^{=}, b^{=}) = 3 - 1 = 2$.

3. $dim(P) = 2 < 3 = dim(R^3)$, so P is not full-dimensional. By (**), P has no interior point.



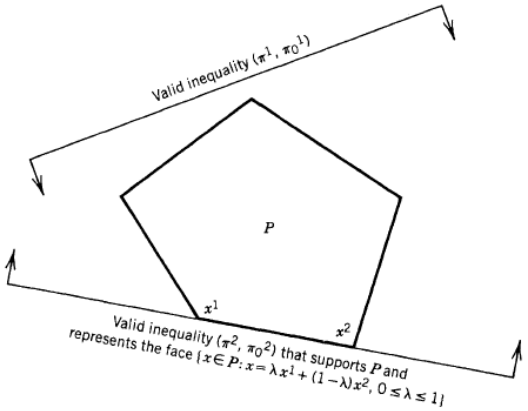
Valid Inequalities

Given a polyhedron, the question we address below is to find out which of the inequalities are necessary in the description of P and which can be dropped.

Definition 2.5.6 The inequality $\pi x \leq \pi_0, or\ (\pi, \pi_0)$, is called a valid inequality for P if it is satisfied by all points in P.

• (π, π_0) is a valid inequality if and only if P lies in the half-space defined $\{x \in R^n : \pi x \leq \pi_0\}$, or equivalently if $\max\{\pi x : x \in P\} \leq \pi_0$

Definition 2.5.7 If (π, π_0) is a valid inequality for P, then $F = \{x \in P : \pi x = \pi_0\}$ is called a **face** of P. We say that (π, π_0) represents F.



• When F is nonempty, we say that (π, π_0) supports P.

Therefore to achieve the necessary description, if an inequality $a_i\ x \leq\ b_i$ is not a support of P we **discard** it.

Definition 2.5.8 A face F is said to be proper if $F\ \neq\ \emptyset$ and $F\ \neq P$. Note that F is nonempty iff $\max\{\pi x: x\ \in\ P\} = \pi_0$

• As $F\ \subseteq\ P$, we have $dim(F)\ \leq\ dim(P)$.
• If F is a proper face of P, then $dim(F) <\ dim(P)$. (***)

Definition 2.5.9 A face F is a **facet** of P if $dim(F)\ =\ dim(P)\ -\ 1$.

• If F is a facet of P, there exists some inequality $a_k x\ \leq\ b_k\ for\ k\ \in\ M^{\leq}$ representing F.

Proposition 2.5.10 If a face F is a **facet** of P, then it is sufficient and necessary to have an inequality representing F in the description of P.

Therefore, every inequality $a_r x\ \leq\ b_r\ for\ r\ \in\ M^{\leq}$ that represents a face of P of dimension less than $dim(P)\ -\ 1$ is irrelevant to the description of P and can be removed.

Remarks
A full-dimensional polyhedron P has a unique minimal representation by a finite set of linear inequalities. In particular, for each facet F_i of $P\ =\ \{x\ \in\ R^n\ :\ Ax\ \leq\ b\}$, there is an inequality $a_i\ x\ \leq\ b_i$ representing F_i.

The description for below example is not unique because P is not full-dimensional.

Example5 (continued)
For the polyhedron P in previous example,
1. Is $(\pi, \pi_0) = (-1, -1,\ 1,\ 1)$ a valid inequality?
2. Is $F_1 = \{x\ \in\ P:\ -x_1 - x_2 + x_3 =\ 1\}$; a face of P? Is it a facet?
3. Is $F_2 = \{x\ \in\ P:\ x_1 + x_3 =\ 1\}$ a facet?
4. Is constraint $x_3 \leq 2$ necessary in the description of P?
5. Is constraint $x_1 + x_2 + 2x_3\ \leq 2$ necessary in the description of P?

Solution
1. (π, π_0) defines the inequality $-x_1 - x_2 + x_3 \leq\ 1$. It is a valid inequality as the whole polyhedron is in the feasible halfspace defined by it.

Check: $\max\{-x_1 - x_2 + x_3 : x \in P\} \leq \pi_0 = 1$. The maximum is taken at $(0,0,1)^T$ having $-x_1 - x_2 + x_3 = 1 \leq 1 = \pi_0$. Note I have also (1,0,0) and (0,1,0) but both of them give -1<1 which means $\pi \neq \pi_0$. So we take max at (0,0,1).

2. By definition $F = \{x \in P : \pi x = \pi_0\}$, with (π, π_0) a valid inequality, is called a face of P. (π, π_0) here is (-1 -1 1, 1) which has just been shown to be a valid inequality in Q1. Therefor, F_1 is a face of P. It is not a facet as $F_1 = \{(0,0,1)^T\}$ contains a single point. So $dim(F_1) = 0 < 1 = dim(P) - 1$. Note if we have 2 points we get a line and its dim is 1. Note also that it is not listed in the description of P.

3. Firstly, F_2 is represented by an inequality in the original constraint set, so it is a face. To check if F_2 is a facet, we need to check its dimension.

Let's consider points satisfying all equations in F_2: $\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + \quad x_3 = 1 \end{cases}$

This system has at least two affinely independent solutions, e.g., $(1,0,0)^T, (0,0,1)^T$ satisfying all other inequality constraints in P, so

$dim(F_2) \geq 1$. Also by (***) we have $dim(F_2) < dim(P) = 2$, so $dim(F_2) \leq 1$. Combine we get $dim(F_2) = 1$. $As\ dim(F_2) = 1 = 2 - 1 = dim(P) - 1$, F_2 is a facet.

4. Define face $F_4 = \{x \in P : x_3 = 2\}$.

Consider points satisfying all equations in F_4: $\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_3 = 2 \end{cases}$

This system has no feasible solution satisfying $x_1, x_2 \geq 0$.

So $F_4 = \emptyset$. F_4 is not a facet, so the inequality $x_3 \leq 2$ is not necessary in the minimal description of P.

Another explanation: consider $(\pi, \pi_0) = (0\ 0\ 1, 2)$ corresponding $x_3 \leq 2$.

$\max\{x_3 : x \in P\} = 1 < \pi_0$. So (π, π_0) is a valid inequality but not supporting P. then $x_3 \leq 2$ can be discarded from P.

5. Define face $F_5 = \{x \in P : x_1 + x_3 + 2x_3 = 2\}$.

Consider points satisfying all equations in F_5: $\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + 2x_3 = 2 \end{cases}$

There is a single solution $(0,0,1)^T$ satisfying all other inequality constraints in P, so $F_5 = \{(0,0,1)^T\}$.

As $dim(F_5) = 0 < 2 - 1 = dim(P) - 1$, F_5 is not a facet and therefore the inequality $x_1 + x_2 + 2x_3 \leq 2$ is not necessary in the minimal description of P.

Another explanation: consider $(\pi, \pi_0) = (1\ 1\ 2, 2)$, it is in the initial description of P. (π, π_0) supports P because $F_5 = \{x \in P: x_1 + x_2 + 2x_3 = 2\}$ is not empty, where $\max\{x \in P: x_1 + x_2 + 2x_3\} = 2 = \pi_0$. The equation set of F_5, i.e the inequalities of P that are satisfied as equalities by F_5 , is:

$\{x_1 + x_2 + 2x_3 = 2$
$x_1 + x_2 + x_3 = 1$
$x_1 + \quad + x_3 = 1$
$-x_1 \quad\quad = 0$
$\quad -x_2 \quad\quad = 0\}$

$$A^{=} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}; \quad b^{=} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$rank(A^{=}, b^{=})$=3; as $dim(F_5) + rank(A^{=}, b^{=}) = n$, then $dim(F_5) + 3 = 3$, $dim(F_5) = 0$.